Received: *May 1, 2000*, Accepted: *June 20, 2000*

The interplay between θ and T

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Abstract: We extend a recent computation of the dependence of the free energy, F, on the non-commutative scale θ to theories with very different UV sensitivity. The temperature dependence of F strongly suggests that a reduced number of degrees of freedom contributes to the free energy in the non-planar sector, F_{np}, at high temperature. This phenomenon seems generic, independent of the UV sensitivity, and can be traced to modes whose thermal wavelengths become smaller than the noncommutativity scale. The temperature dependence of F_{np} can then be calculated at high temperature using classical statistical mechanics, without encountering a UV catastrophe even in large number of dimensions. This result is a telltale sign of the low number of degrees of freedom contributing to F in the non-planar sector at high temperature. Such behavior is in marked contrast to what would happen in a field theory with a random set of higher derivative interactions.

Keywords: Bosonic Strings, Space-Time Symmetries.

JHEP06(2000)032

Contents

1. Introduction

Non-commutative field theories have proven to exhibit fascinating properties ([1]–[24]).

As we noted in a recent paper [7], there is a drastic reduction in the number of degrees of freedom that contribute to the free energy in the non-planar sector, $F_{\rm np}$, at high temperatures. By the free energy in the non-planar sector, we mean those contributions to the free energy which arise from non-planar Feynman graphs. This reduction can be read off by looking at the temperature dependence of $F_{\rm np}$. In this paper, we will present further evidence for this reduction, and study two further theories with very different UV sensitivity to test how generic this high temperature behavior is.

We will take $\theta_{0i} = 0$ throughout.

In [7], we also noted the existence of winding states in non-commutative field theories. Though we have no clean way of treating these modes separately from the conventional degrees of freedom of commutative field theory, it appears plausible that $F_{\rm np}$ is indicative of these additional degrees of freedom, especially at high temperature, at which the winding states become light.

In studying the reduction of the degrees of freedom that contribute to $F_{\rm np}$, we emphasize that this phenomenon does not depend on the sensitivity of the corresponding commutative theory to the ultraviolet [6]. Indeed, this reduction occurs in theories that are extremely insensitive to the ultraviolet, like the $N = 4$, $D = 4$ SYM theory, as well as theories with strong ultraviolet sensitivity, as e.g. $\lambda\phi^3$ in $D = 6$.

In a nutshell, what we think happens is that as the temperature rises above the noncommutativity scale, the modes of the various fields with momenta of order the temperature, T, do not contribute to $F_{\rm np}$. This occurs because, as the thermal wavelengths drop below the noncommutativity scale $\theta^{1/2}$, as far as their contribution to $F_{\rm np}$ is concerned, there is no way to distinguish these high momenta modes from each other and thus no way to count their contribution individually. Another way of thinking about this is that the contributions to $F_{\rm np}$ are sensitive to the phase space structure of space: the uncertainty principle among the spatial coordinates x_i renders the identification of the degrees of freedom with regard to their contribution to $F_{\rm np}$ impossible; this contribution seems to be summarized by a few quantum degrees of freedom contributing per non-commutative cell whose area is θ. In the following, we will refer to this cell as the Moyal cell.

In section 2, we briefly review the results for the free energy of the Wess-Zumino model. We introduce a tool to test whether a reduction of degrees of freedom contributing to $F_{\rm np}$ in fact takes place, the calculation of the classical statistical mechanics approximation to F, and discuss when we expect this approximation to be valid.

In section 3, we discuss the case of $N = 4$, $D = 4$ SYM. This theory also shows a reduction of degrees of freedom at high temperatures, although the theory is rather insensitive to the UV. We see that the non-planar sector again contains winding states.

In section 4, we show how these results generalize to higher dimensions where there can be more non-commutative spatial directions. We show that the classical approximation remains valid for $\lambda\phi^3$ in spite of the high sensitivity of the corresponding commutative theory to the UV.

Finally, we end with conclusions.

2. The thermodynamics of the non-commutative Wess-Zumino model in perturbation theory

The lagrangian density for the Wess-Zumino model is:

$$\mathcal{L} = i\,\partial_\mu\bar\psi\,\bar\sigma^\mu\,\psi + A^*\Box\,A - \frac{1}{2}\,M\,\psi\,\psi - \frac{1}{2}\,M\bar\psi\,\bar\psi - g\,\psi * \psi\,A - g\,\bar\psi * \bar\psi\,A^* - F^*F\,,$$

where F is given by

$$F = -M\,A^* - g\,A^* * A^*\,.$$

We showed in a previous paper that in the non-commutative Wess-Zumino model, one can distinguish two regimes of high temperature, $\beta M \ll 1$.[1] These two regimes are: $\theta T^2 \ll 1$ and $\theta T^2 \gg 1$.

In the case $\theta T^2 \ll 1$, the thermal wavelength is larger than the noncommutativity scale. Here, $F_{\rm np}$ behaves as

$$\left.\frac{F}{V}\right|_{\rm np} \sim -g^2\, T^4\,. \tag{2.1}$$

In the other limit, the free energy density behaves as

$$\left.\frac{F}{V}\right|_{\rm np} \sim -g^2\,\frac{T^2}{\theta}\,\log T^2\,\theta\,. \tag{2.2}$$

This expression for the free energy shows a dramatic reduction of the degrees of freedom contributing to $F_{\rm np}$.

Such a reduction has a natural interpretation in terms of the novel phase space structure due to the non-commutative nature of space. At any temperature, the typical momenta of the fields have magnitudes of order the temperature or smaller. As the temperature rises, the thermal wavelengths become smaller. When the temperature reaches values of $O(\theta^{-1/2})$, modes of the field with momenta of $O(T)$ no longer contribute to $F_{\rm np}$.

More precisely, what we think happens is that at temperatures $T^2 \gg 1/\theta$, the modes ϕ_{k_1,k_2,k_3} with $k_{1,2} \geq \theta^{-1/2}$ cannot be distinguished from each other insofar as their contribution to $F_{\rm np}$ is concerned, and therefore will not be counted individually in the non-planar contribution to the partition function. These modes lose their individual identity in the non-planar sector because the noncommutativity between x_1 and x_2, $[x_1,x_2] \neq 0$, implies an uncertainty relationship which renders impossible their separate identification. This may explain the reduction in the temperature dependence of the free energy from T^4 to T^2/θ.

The non-planar sector yields a contribution to the free energy which, could this sector be isolated, would suggest that it has the number of degrees of freedom of a (1+1)-dimensional field theory at temperatures $T \gg 1/\theta^{1/2}$. This is reminiscent of the result obtained by Atick and Witten [25] in the context of string theory. There is also a subleading, logarithmic dependence on the temperature in eq. (2.2) for the free energy, which is all that remains from the contributions of the high momenta components of the fields.

As announced in the introduction, one can compare this computation to a classical statistical mechanics calculation of the free energy. The classical calculation can

[1] We will assume that the Compton wavelength of the fields is bigger than the non-commutative scale, $\theta M^2 \ll 1$.

be done by just keeping the zero frequency term in the sum over frequencies. The result is

$$\left.\frac{F}{V}\right|_{\mathrm{np}} \sim -g^2\,\frac{T^2}{\theta}\,\log\Lambda\,, \tag{2.3}$$

where Λ is an ultraviolet cutoff.

The classical calculation captures the correct power law dependence in the temperature but replaces the logarithmic dependence on temperature obtained in quantum mechanics by $\log\Lambda$.

We should remind the reader that classical statistical mechanics is a good approximation at high temperatures for the free energy of systems with few degrees of freedom. For example, the high temperature behavior of the free energy for a particle of mass m in a potential $V(x)$ is well approximated by classical statistical mechanics when the thermal de-Broglie wavelength $\lambda_{\mathrm{DB}} \sim 1/(mT)^{1/2}$ is smaller than the length scale over which the potential varies. In the path integral formulation, this would correspond to actually shrinking the temporal circle down to zero size and performing a dimensional reduction. In Feynman diagrams, this amounts to only keeping the contributions of the zero frequency components in the sum over frequencies.

In contrast, if one uses classical statistical mechanics to evaluate the free energy of systems with a field theory number of degrees of freedom at high temperature, one inevitably encounters a UV catastrophe.

On the other hand, classical statistical mechanics is a good approximation in field theory when calculating thermal correlation functions. An example is the calculation of the two point correlation function between two operators, $O_1(x)$ and $O_2(y)$,

$$\Big\langle O_1(x)\,O_2(y)\Big\rangle - \langle O_1(x)\,\rangle\,\langle O_2(y)\rangle\,.$$

Indeed, the modes that primarily contribute to this correlation function have wavelength commensurate with the distance $|x-y|$ separating the probes. As one heats the system, the population of these modes increases and hence their contribution to the correlation function is well approximated by classical statistical mechanics. So the classical approximation at high temperature becomes applicable in field theory when there is a bound on the wavelengths of the modes that contribute to the quantity we are calculating.

The fact that F_{np} is well approximated by classical statistical mechanics thus suggests that only a reduced set of degrees of freedom contributes to F_{np} at high temperatures. This should be contrasted to the cases of field theories with an arbitrary infinite series of higher derivative terms in the lagrangian. In such generic cases, if one attempts to approximate the free energy by a classical statistical calculation, one encounters enhanced UV catastrophes as compared to the theory without the higher derivative terms. The averting of the UV catastrophe due to the special choice of higher derivative terms in non-commutative field theories is therefore quite remarkable.

3. The thermodynamics of $N=4,\ D=4$ Super Yang-Mills

Our interest in examining this case is to see how crucial the ultraviolet sensitivity of a non-commutative theory is to the existence of winding states and the high temperature behavior of $F_{\rm np}$. We will show that the existence of winding states as well as the reduction at high temperatures in the number of degrees of freedom contributing to $F_{\rm np}$ are independent of the UV behavior of the non-commutative field theory.

The calculation of the free energy is straightforward, the result can be found in the literature [21, 10]. We present for completeness some of the steps in the calculation of the free energy in appendix A.

The contribution to the free energy coming from the non-planar sector is:

$$\left.\frac{F}{V}\right|_{\rm np} = -4\,g^2\,N\int\frac{d^3p}{(2\pi)^3}\,\frac{d^3k}{(2\pi)^3}\,\frac{e^{i\,p\,\theta\,k}}{4\,\omega_p\,\omega_k}\Big(n_B\left(\omega_p\right)+n_F\left(\omega_p\right)\Big)\times \\ \times\Big(n_B\left(\omega_k\right)+n_F\left(\omega_k\right)\Big)\,. \tag{3.1}$$

Up to an overall factor, this is exactly the result we found in the Wess-Zumino case [7].

The presence of winding states can again be detected in the non-planar sector. Indeed, by performing the integration over one set of momenta in eq. (3.1), one finds contributions to the free energy that are weighted by the length of the temporal circle. This can be seen for example by considering the following factor in eq. (3.1):

$$\int\frac{d^3k}{(2\pi)^3}\,\frac{e^{i\,p\,\theta\,k}}{\omega_k}\,n_B\left(\omega_k\right)=\sum_{n=1}^{\infty}\frac{1}{n^2\,\beta^2+(\theta\,p)^2}\,. \tag{3.2}$$

We can again compare the limits $T^2\theta \ll 1$ and $T^2\theta \gg 1$. In the case where the thermal wavelength is larger than the noncommutativity scale, the free energy behavior is:

$$\left.\frac{F}{V}\right|_{\rm np}\sim -g^2\,N\,T^4\,. \tag{3.3}$$

In the limit where the thermal wavelength is within the non-commmutative scale,

$$\left.\frac{F}{V}\right|_{\rm np}\sim -g^2\,N\,\frac{T^2}{\theta}\log\left(T^2\,\theta\right)\,. \tag{3.4}$$

This reveals again a reduction in the number of degrees of freedom contributing to $F_{\rm np}$.

There seems to be a universal behavior that whenever the inverse momenta fit into a Moyal cell, the associated components of the field cannot be distinguished in their contributions to $F_{\rm np}$. What is then left over seems to be the contribution due to a few quantum degrees of freedom per Moyal cell.

This picture will be further tested in the next section, where we will consider a higher-dimensional example.

JHEP06(2000)032

4. The thermodynamics of $g\phi^3$ in $D = 6$ dimensions

Strictly speaking, this theory does not have good thermodynamic behavior since it lacks a ground state. Therefore, in order to discuss the thermodynamics of this system, we will take the coupling constant to be very small and the temperature to be small enough[2] such that the excursions of the field ϕ away from the local minimum of the potential are within the bounded region of the potential. This implies that the temperature satisfies the inequality $T \ll M/g^{1/2}$.

The lagrangian density for this system is:

$$\mathcal{L} = \int d^6x \Big((\partial\phi)^2 - M^2\,\phi^2 - g\,\phi * \phi * \phi \Big)\,.$$

The free energy in this case has no infrared divergences to leading order in an expansion in the coupling constant g. This is because the high dimensionality of the theory softens the IR sensitivity of the free energy.

The non-planar contribution to the free energy at $O(g^2)$ is:

$$\frac{F}{V}\bigg|_{\mathrm{np}} = -g^2\,T^2 \sum_{n,l} \int \frac{d^5p}{(2\pi)^5}\,\frac{d^5k}{(2\pi)^5} \times \\ \times \frac{e^{i\,p\,\theta\,k}}{\left(\frac{4\pi^2n^2}{\beta^2} + k^2\right)\left(\frac{4\pi^2l^2}{\beta^2} + p^2\right)\left(\frac{4\pi^2(n+l)^2}{\beta^2} + (k+p)^2\right)}\,. \tag{4.1}$$

Again, one finds winding states in this theory. This can easily be seen by rewriting eq. (4.1) in the form:[3]

$$\frac{F}{V}\bigg|_{\mathrm{np}} = -g^2\,T \sum_{n,l} \int \frac{d^5p}{(2\pi)^5} \int_0^1 dx \int_0^\infty \frac{d\alpha_1}{\alpha_1^2}\,d\alpha_2\,e^{-(\alpha_2+\alpha_1 x(1-x))(p^2+\frac{4\pi^2n^2}{\beta^2})} \times \\ \times\, e^{-\frac{(\theta p)^2+4\pi^2l^2\beta^2}{\alpha_1}}\,e^{2\pi inlx}\,, \tag{4.2}$$

where we can think of $\alpha_2 + \alpha_1 x(1-x)$ as the proper time for the propagator of momentum states and $1/\alpha_1$ as the proper time for the propagator of winding states.

Under the restriction that $\theta_{0i} = 0$, the most general case in $D = 6$ is, by a convenient choice of coordinate system, given by non-vanishing θ_{12} and θ_{34}. If we take these two parameters to be of the same order of magnitude, one can distinguish two cases:

1. $T^2 \ll 1/\theta_{12}, 1/\theta_{34}$.
2. $T^2 \gg 1/\theta_{12}, 1/\theta_{34}$.

[2]while still keeping it much larger than the mass.
[3]for details, see appendix B.

In the first case, the contribution to the free energy from the non-planar sector scales, as a function of temperature, like T^6. This is because the exponential involving the Moyal phase does not oscillate much when the momenta are distributed according to the thermal distributions.

In the high temperature limit where $T^2 \gg 1/\theta_{12}, 1/\theta_{34}$, the possiblity of estimating the behavior of the free energy using classical physics arises. Indeed, as discussed above, if the system has many fewer degrees of freedom contributing to the θ dependence of the free energy, as was the case in the 4-dimensional examples, then this approximation is valid.

Performing the classical statistical mechanics calculation, i.e. evaluating the following expression:

$$\left.\frac{F}{V}\right|_{\rm np} \sim -g^2\, T^2 \int \frac{d^5k}{(2\pi)^5}\, \frac{d^5p}{(2\pi)^5}\, \frac{e^{i\,p\,\theta\,k}}{p^2\, k^2 (p+k)^2}\,, \tag{4.3}$$

we find no UV divergences, an indication that we are dealing with fewer degrees of freedom than the dimension of the system would naively suggest.

When $\theta \sim \theta_{12} \sim \theta_{34}$,

$$\left.\frac{F}{V}\right|_{\rm np} \sim -g^2\, \frac{T^2}{{\theta_{34}}^2 - {\theta_{12}}^2} \log \frac{\theta_{12}}{\theta_{34}} \sim -\frac{g^2\, T^2}{\theta^2}\,. \tag{4.4}$$

To establish that T^2 is in fact the leading temperature behavior of the free energy in the non-planar sector, we prove the finiteness of the sum (4.1) and then note that every contribution to the sum other than from $n = l = 0$ grows more slowly than quadratically in T, see appendix B for details.

The T^2 behavior is again consistent with the picture on how degrees of freedom, as their inverse momenta fall into Moyal cells, do not contribute to $F_{\rm np}$.

5. Conclusions

At high temperatures, one observes a substantial reduction in the number of degrees of freedom that contribute to $F_{\rm np}$, the free energy due to the non-planar sector of the theory. The picture that emerges from the previous sections is that this phenomenon can be traced to the modes of the fields with momenta larger than the noncommutativity scales. What happens is that once the wavelengths are within the Moyal cells, there is no way to distinguish and count the separate contributions of these modes to $F_{\rm np}$. What is left over seems to be the contribution of a single degree of freedom per Moyal cell.

Because of this severe reduction in the number of degrees of freedom contributing to $F_{\rm np}$, this quantity can be calculated at high temperature using classical statistical mechanics.

This behavior does not appear to depend on the details of the non-commutative field theory, the crucial element is the existence of a phase space structure for space.

JHEP06(2000)032

Acknowledgments

We thank Nathan Seiberg for very useful discussions. The work of WF, EG, RMcN, AK-P, SP, PP is supported in part by the Robert Welch Foundation and the NSF under grant number PHY-9511632. SP is also supported by NSF grant PHY-9973543.

A. The free energy of $N=4$, $D=4$ SYM

We are working in euclidean spacetime.

Let's begin with the commutative case. We choose the decomposition of the 32-dimensional Γ-matrices given in [26]. This yields the dimensionally reduced lagrangian

$$\begin{aligned}\mathcal{L} = 2\,\mathrm{Tr}\Big(&-\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{i}{2}\,\bar{\chi}_K\,\bar{\sigma}^\mu\,D_\mu\,\chi_K - \frac{1}{2}\,D_\mu\,\phi_i\,D^\mu\phi_i - \frac{1}{2}\,D_\mu\,\varphi_i\,D^\mu\,\varphi_i - \\ &- \frac{i}{2}\,g\,\alpha^i_{KL}\Big(\chi_K\,[\phi_i,\chi_L] + \bar{\chi}_K\,[\phi_i,\bar{\chi}_L]\Big) - \\ &- \frac{i}{2}\,g\,\beta^i_{KL}\Big(-\chi_K\,[\varphi_i,\chi_L] + \bar{\chi}_K\,[\varphi_i,\bar{\chi}_L]\Big) - \\ &- \frac{1}{4}\,g^2\Big([\phi_i,\phi_j]\,[\phi_i,\phi_j] + [\varphi_i,\varphi_j]\,[\varphi_i,\varphi_j] + 2\,[\phi_i,\varphi_j]\,[\phi_i,\varphi_j]\Big)\Big)\,, \end{aligned} \tag{A.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,g[A_\mu, A_\nu]$, $D_\mu = \partial_\mu + i\,g[A_\mu,\cdot]$ and μ, ν range from 1 to 4. Remember that this theory contains one $N=1$ vector multiplet and three $N=1$ chiral multiplets. Thus i ranges from 1 to 3, K and L from 1 to 4. γ_μ, α_i and β_i satisfy $\{\gamma_\mu,\gamma_\nu\} = -2\,\delta_{\mu\nu}$, $\{\alpha_i,\alpha_j\} = -2\,\delta_{ij}$, $\{\beta_i,\beta_j\} = -2\,\delta_{ij}$ and commute among each other.

The diagrams that contribute to the free energy at two loop are depicted in figure 1 below.

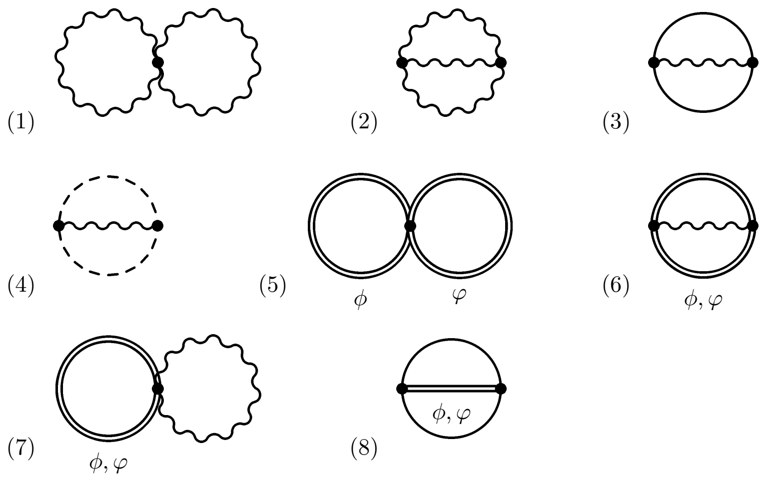

Figure 1: Diagrams that contribute to the free energy at two loops.

At $T=0$, their contribution must vanish. It is easy to see that each diagram is proportional to

$$g^2 f_{abc} f_{abc} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{1}{k^2p^2}\,.$$

The coefficients of the various contributions are given in the following table:

1	2	3	4	5	6	7	8
-3	$9/4$	4	$-1/4$	$9/2$	$9/2$	-12	0

(A.2)

We pass to finite temperature by replacing the integration over energies by a sum over even or odd Matsubara frequencies, for bosonic, fermionic degrees of freedom, respectively.

The non-commutative case does not require any additional calculation. We obtain the non-commutative theory by replacing the commutators in the lagrangian (A.1) by Moyal brackets. This yields, for arbitrary fields A, B, which take values in the fundamental representation,

$$[A,B]_*(x) = \int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \tilde{A}^a(p)\, \tilde{B}^b(k)\, e^{i(p+k)x} \left(\left[t^a, t^b\right] \cos\frac{k\theta p}{2} + i\left\{t^a, t^b\right\} \sin\frac{k\theta p}{2} \right).$$

For the gauge group $\mathrm{U}(N)$, both the commutator and the anticommutator of generators of the fundamental close within the algebra:

$$\left[t^a, t^b\right] = i\, f_{abc}\, t^c\,, \quad \left\{t^a, t^b\right\} = d_{abc}\, t^c\,.$$

The non-commutative result is thus obtained from the commutative one by replacing [10]

$$f_{abc} \longrightarrow f_{abc} \cos\frac{k\,\theta\, p}{2} + d_{abc} \sin\frac{k\,\theta\, p}{2}\,.$$

To obtain (3.1), we need to perform the sums $f_{abc}f_{abc}$ and $d_{abc}d_{abc}$. The first is of course identical to the $\mathrm{SU}(N)$ result, $N(N^2-1)$. The second can be obtained from the $\mathrm{SU}(N)$ result, $N(N-\frac{4}{N})$, where $\{t^a, t^b\} = d^{\mathrm{SU}(N)}_{abc} t^c + \frac{1}{N}\delta_{ab}$, by including the $\mathrm{U}(1)$ generator $t^{N^2} = \frac{1}{\sqrt{2N}}1$. This gives $N(N^2+1)$.

B. The free energy of the $D=6$, $g\phi^3$ theory

We will first briefly show how the winding states appear in this case, then we will perform the classical statistical mechanics calculation of the non-planar contribution to $O(g^2)$ to the free energy.

The non-planar contribution to the free energy to $O(g^2)$ is

$$-g^2\, T^2 \sum_{n,l} \int \frac{d^5p}{(2\pi)^5} \frac{d^5k}{(2\pi)^5} \frac{e^{i(\theta_{12}(p_1\, k_2 - p_2\, k_1) + \theta_{34}(p_3\, k_4 - p_4\, k_3))}}{\left(p^2 + \frac{4\pi^2 n^2}{\beta^2}\right)\left(k^2 + \frac{4\pi^2 l^2}{\beta^2}\right)\left((p+k)^2 + \frac{4\pi^2 (n+l)^2}{\beta^2}\right)}\,. \quad \text{(B.1)}$$

The appearance of winding contributions in this integral can be shown by introducing a Feynman parameter, x, and Schwinger parameters α_1 and α_2:

$$-g^2\,T^2\sum_{n,l}\int\frac{d^5p}{(2\pi)^5}\,\frac{d^5k}{(2\pi)^5}\int_0^1 dx\int_0^\infty d\alpha_1 d\alpha_2\,\alpha_1 e^{-\alpha_2(p^2+\frac{4\pi^2n^2}{\beta^2})}\times$$
$$\times\, e^{-\alpha_1 k^2-\alpha_1(p^2x+2pkx+4\pi^2\frac{n^2x+2nlx+l^2}{\beta^2})}e^{i\,p\,\theta\,k}\,. \tag{B.2}$$

Performing the integral over k followed by a Poisson resummation over l gives

$$\left.\frac{F}{V}\right|_{\rm np}=-g^2\,T\sum_{n,l}\int\frac{d^5p}{(2\pi)^5}\int_0^1 dx\int_0^\infty\frac{d\alpha_1}{\alpha_1^2}\,d\alpha_2\,e^{-(\alpha_2+\alpha_1x(1-x))(p^2+\frac{4\pi^2n^2}{\beta^2})}\times$$
$$\times\, e^{-\frac{(\theta p)^2+4\pi^2l^2\beta^2}{\alpha_1}}e^{2\pi inlx}\,, \tag{B.3}$$

which is eq. (4.2).

To prove the convergence of (B.1), we introduce Feynman and Schwinger parameters and perform some gaussian integrals to obtain

$$-g^2\,T^2\sum_{n,l=-\infty}^{\infty}\int_0^\infty\alpha\,d\alpha\int_0^1 dx\,\times$$
$$\times\int_0^{1-x}dy\,\frac{e^{-\frac{4\pi^2}{\beta^2}\alpha(n^2y+l^2(1-x-y)+(n+l)^2x)}}{[f(x,y)^{1/2}]\,[\alpha^2f(x,y)+\theta_{12}^2]\,[\alpha^2f(x,y)+\theta_{34}^2]}\,, \tag{B.4}$$

where $f(x,y)=x(1-x)+y(1-y)-x\,y$. Ignoring the coefficient in front of this integral, notice that the summands are positive and monotonously decreasing in n and l. We can thus apply the integral test, which states that the sum over n and l is convergent iff the integral is. The latter however can be computed to yield:

$$-g^2\,T^2\int_{-\infty}^{\infty}dn\,dl\int_0^\infty\alpha\,d\alpha\,\times$$
$$\times\int_0^1 dx\int_0^{1-x}dy\frac{e^{-\frac{4\pi^2}{\beta^2}\alpha(n^2y+l^2(1-x-y)+(n+l)^2x)}}{[f(x,y)^{1/2}]\,[\alpha^2f(x,y)+\theta_{12}^2]\,[\alpha^2f(x,y)+\theta_{34}^2]}=$$
$$=-\frac{g^2}{4}\,\frac{1}{\theta_{12}+\theta_{34}}\,\frac{1}{\theta_{12}\,\theta_{34}}\,. \tag{B.5}$$

The convergence of this result (which incidently yields the zero temperature vacuum energy of the theory) hence proves the finiteness of (B.1).

From the T-dependence of the exponential in (B.4), we can immediately see that the leading contribution at large T comes from the $n=l=0$ contribution, yielding

$$-g^2\,T^2\,\frac{1}{\theta_{12}^2-\theta_{34}^2}\,\log\left(\frac{\theta_{12}}{\theta_{34}}\right). \tag{B.6}$$

This is the general result when $M/g^{1/2}\gg T\gg M$.

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JHEP06(2000)032

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